# 1 Shot Noise

## 1.1 History and Background

Shot noise is due to the corpuscular nature of transport. In 1918, Walter Schottky discovered Shot noise in tubes and developed Schottky's theorem. Shot noise is *always* associated with direct current flow. In fact, it is required that there be dc current flow or there is **no** Shot noise. Electrical currents do not flow uniformly and do not vary smoothly in time like the standard water flow analogy. Current flow is not continuous, but results from the motion of charged particles (i.e. electrons and/or holes) which are discrete and independent. At some (supposedly small, presumed microscopic) level, currents vary in unpredictable ways. It is this unpredictable variation that is called *noise*.

If you could "observe" carriers passing a point in a conductor for some time interval you would find that a "few" more or less carriers would pass in one time interval versus the next. It is impossible to predict the motion of individual electrons, but it is possible to calculate the average net velocity of an ensemble of electrons, or the average number of electrons drifting past a particular point per time interval. The variation about the mean value or average of these quantities is the noise. In order to "see" Shot Noise, the carriers must be constrained to flow past in one direction only. The carrier entering the "observation" point must do so as a purely random event and independent of any other carrier crossing this point. If the carriers are not constrained in this manner then the resultant thermal noise will dominate and the Shot Noise will not be "seen". A physical system where this constraint holds is a *pn* junction. The passage of each carrier across the depletion region of the junction is a random event, and because of the energy barrier the carrier may travel in only one direction. Since the events are random and independent, Poisson statistics describe this process.

To try and find the statistics of this process will require a physical model to analyze, therefore we will consider an $LC$ tank circuit.

## 1.2 Derivation of Shot Noise

As an illustration think about an $LC$ tank circuit being charged through an ideal switch (such as an ideal *pn* junction diode) from a battery with a voltage $V$. Now the switch is capable of turning on and off in such a short time interval that only single electrons pass through to the tank circuit. Therefore the current pulse is negligible. Also the switch randomly turns on and off such that the current pulses are independent and uncorrelated. Then we can approximate the current flowing into the tank circuit as a spike of current or delta function

$$I(t) = \sum_j q\,\delta(t - t_j)\ , \tag{1}$$

where the $t_j$'s are the random arrival times of the electrons. What we know about $t_j$ is that on average there should be $\bar{I}/q$ of them per unit of time following the definition of current.

Now, what does one of these current pulses do to the $LC$ circuit? A short pulse of current is not going to go through the inductor, so it must end up charging the capacitor. This will produce a rapid change in the voltage on the capacitor whose magnitude is

$$\delta V = q/C\ . \tag{2}$$

If initially the $LC$ circuit contained no energy (i.e. voltage and current identically zero), the first pulse of current would start the circuit oscillating with a voltage amplitude of $\delta V$. Subsequent pulses of current would arrive at unpredictable times within the period of oscillation, so that some pulses might increase the amplitude of the oscillation and others might decrease the amplitude. Let us write the voltage on $C$ as

$$V(t) = \Re\left[V_a \exp\left(i\,t/\sqrt{LC}\right)\right]\ , \tag{3}$$

where $V_a$ is the complex-valued amplitude of the oscillation. Then the effect of the arrival of a current pulse is to translate $V_a$ in the complex plane by a distance $\delta V$ but in a random direction. Thus, $V_a$ will execute a *random walk* in the complex plane. For the present purposes, the significant feature of a random walk is that the average value of $V_a$ is zero, but the actual value is almost never zero. That is, the spikes in the current through the switch will keep the $LC$ tank circuit excited to some level. This is a characteristic feature of noise.

If you are familiar with the theory of random walks, you will have noticed that $|V_a|$ can grow without limit. This unphysical result is due to our having neglected any possibility of back-action of the $LC$ circuit on the ideal switch/battery system. However this is necessary for the physics of Shot noise since the electrons must be restricted to travel in one direction only and not retrace their path. If the expectation of the electron was equally likely to go either direction then Thermal noise would dominate.

Now lets us suppose that the switch will close for a long time and open for a short time, so that the current pulses are of a long duration compared to the resonant frequency of the $LC$ circuit. Such pulses are not going to excite much of an oscillation in that circuit. Thus, there is a significant difference in the properties of the Shot noise, depending upon the duration of the current pulses. This is clearly a property of the noise-producing component, rather than of the $LC$ tank circuit. The underlying concept is that the noise is distributed over a spectrum of frequencies, and the form of the distribution function, or *noise spectrum* is the key property.

A physical switch that has this property is a *pn* junction diode. It is well known that semiconductor diodes exhibit Shot noise. This is because the built-in potential across the depletion layer of the *pn* junction is high enough to prevent the majority carriers from returning once they cross the junction. The transit time across the depletion region is the key time constant for the diode and the carrier arrival is an independent and random event.

We will examine the mathematical machinery by which one evaluates the noise spectrum, and then apply it to find Shot and Thermal noise. The mathematical object which allows us to characterize the duration of the current pulse is called the *autocorrelation function* and is defined by

$$R_I(t') = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I(t)I(t+t')\, dt \ . \tag{4}$$

The Wiener-Khintchine theorem states that the noise spectrum is the Fourier transform of the autocorrelation function:

$$S_I(f) = 2\int_{-\infty}^{\infty} R_I(t')\, e^{-i\,2\pi f t'}\, dt' \ , \tag{5}$$

where $S_I(f)$ is the one-sided power spectral density (PSD) and physically for this case is the mean-square current fluctuation in a unity bandwidth, $S_I(f) = \overline{i^2}/\Delta f$. These definitions follow from the facts that only real, positive frequencies are used in circuit analysis and that for a noise process the mean is always zero so that the variance is equal to the mean-square value. The two-sided spectral density is an even function of frequency so that $\mathcal{S}(f) = \mathcal{S}(-f)$ which leads to the fact that $S(f) = 2\mathcal{S}(f)$, for $f \geq 0$ and provides the factor of 2 in the Fourier transform (5).

Now, we apply (4) to (1) to find the autocorrelation of delta-function current pulses.

$$\begin{aligned} R_I(t') &= \lim_{T\to\infty} \frac{q^2}{T} \sum_k \sum_{k'} \int_{-T/2}^{T/2} \delta(t-t_k)\delta(t-t_{k'}+t')\, dt \ , \\ &= \lim_{T\to\infty} \frac{q^2}{T} \sum_k \sum_{k'} \delta(t_k - t_{k'} + t') \ , \end{aligned} \tag{6}$$

where the properties of the delta function are used to evaluate the integral. Now, consider the terms in the double summation above. In the case where the summation indices are $k = k'$ which means the arrival times are equal $t_k = t_{k'}$, we just have $\delta(t')$, and if there are $N$ values of $t_k$ such that $-T/2 < t_k < T/2$, these terms will contribute $N\delta(t')$ to the autocorrelation function. For $t_k \neq t_{k'}$, the delta functions will occur at randomly distributed, nonzero values of $t'$. We argue that, with suitable averaging, the contributions from these delta functions to the Fourier transform in (5) will vanish. (Note, however, that entire textbooks have been written on the details that are hidden in the phrase "suitable averaging.") So, the part of the autocorrelation function which remains is given by

$$R_I(t') = q\bar{I}\,\delta(t') \;, \tag{7}$$

where we have used $N/T = \bar{I}/q$ with $\bar{I}$ being the dc current.

Taking the Fourier transform (5): ( $\mathcal{F}\{\delta(t)\} \leftrightarrow 1$ ); we find Schottky's theorem

$$S_I(f) = 2q\bar{I} \;. \tag{8}$$

The spectrum is uniform and extends to all frequencies. This kind of spectrum is called *white* and many textbooks use the symbol $S_I(0)$ to mean no frequency dependence.

Now, let us consider the case for the current pulses being of significant duration. In particular, suppose

$$I(t) = \sum_k q\,s(t - t_k) \;, \tag{9}$$

where $s(t)$ is a square current pulse of duration $\tau$, as illustrated in Figure 1. The auto-

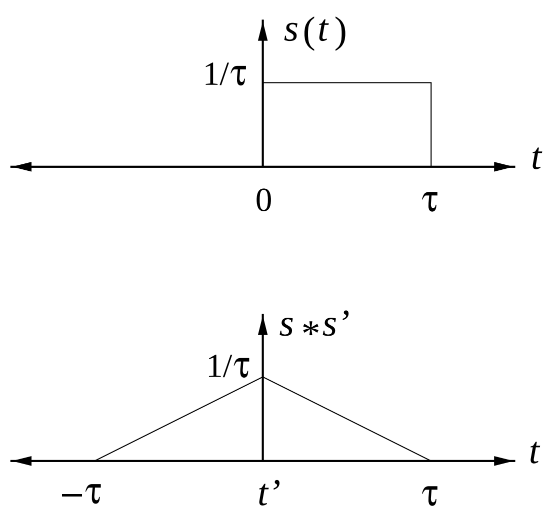


Figure 1: A square current pulse of duration $\tau$ and its autocorrelation function

correlation function can be found by a similar argument to that which we made for the delta-function case earlier and leads to

$$R_I(t') = q\bar{I}\,s(t) * s(t + t') \;, \tag{10}$$

where $s * s'$ is the autocorrelation function of $s$, and is shown in Figure 1. The Fourier transform of a single triangle is:

$$\mathcal{F}\left\{ \begin{array}{rcl} 1-|t'|/\tau & : & |t'| \leq \tau \\ 0 & : & |t'| > \tau \end{array} \right\} \leftrightarrow \tau \left[ \frac{\sin(\pi f \tau)}{\pi f \tau} \right]^2 .$$

When the pulses repeat, as is our case, then the transform is scaled by $1/\tau$. At this point we need to remember that the Fourier transform of a repeating series of pulses with constant period and constant pulse-width, leads to a discrete spectrum with an average or dc value of $\bar{I}$ and harmonics at $n/\tau$; $n = 1, 2, \ldots$. There would be *no noise produced* from dc out to $1/\tau$. This is clearly wrong! The key to finding the right answer is that the electrons arrive at random times (i.e. random periods) and with random transit times (i.e. random $\tau$). Then we should take the Fourier transform of all the possible cases and ensemble average the results. However it is easier to ensemble average the $\tau$'s $\rightarrow \bar{\tau}$ and then apply the Fourier transform. This is equivalent since the Fourier transform is a linear operation. Now applying (5) yields

$$S_I(f) = 2q\bar{I} \left[ \frac{\sin(\pi f \bar{\tau})}{\pi f \bar{\tau}} \right]^2 . \quad (11)$$

This equation gives the same result as (8) at low frequencies, but has a cutoff at $f = 1/\bar{\tau}$ as shown in Figure 2.

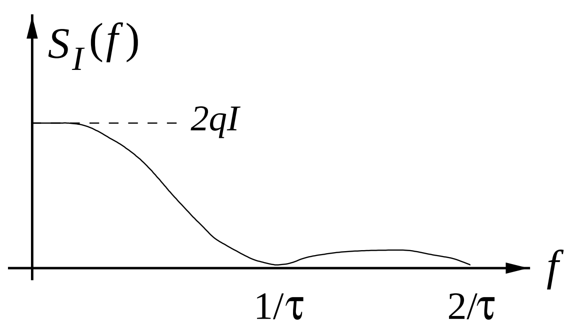


Figure 2: The frequency spectrum of the autocorrelation function for Shot noise

This figure of the spectrum is exaggerated because $\bar{\tau}$ is the mean transit time of the electrons crossing the depletion region of the diode ($\approx$ 10 ps), which means the cutoff frequency is about 100 GHz. Therefore the approximate equation (8) which implies that Shot noise is independent of frequency is good for almost all integrated circuit design. While the frequency distribution of $S_I(0) = 2q\bar{I}$ is *white*, the amplitude distribution is Gaussian due to the Central Limit Theorem (random walks using a very large number of steps).

## 1.3 van der Ziel's Derivation of Shot Noise

To find the fluctuation, first define $N$ as the number of carriers passing a point in a time interval $\tau$ at a rate $n(t)$. Then

$$N = \int_0^\tau n(t)\, dt \ , \tag{12}$$

$$\bar{N} = \bar{n}\,\tau \ , \tag{13}$$

where $\bar{N}$ and $\bar{n}$ are ensemble averages and this results follows from the fact that time averages equal ensemble averages (the Ergodic theorem). If we define a new random variable $\Delta N$ such that

$$\Delta N = N - \bar{N} \ , \tag{14}$$

then we have removed the "d.c." term leaving only the fluctuation. If we also define the random process $X_\tau$ then for sufficiently large $\tau$ we have

$$X_\tau = \frac{\Delta N}{\tau} \ , \tag{15}$$

Note that for a Poisson process $\bar{N} = \mathrm{var}\, N = \overline{\Delta N^2}$ and $\bar{n} = \mathrm{var}\, n$, therefore

$$\overline{X_\tau^2} = \frac{\overline{\Delta N^2}}{\tau^2} = \frac{\mathrm{var}\, N}{\tau^2} = \frac{\bar{N}}{\tau^2} = \frac{\bar{n}\tau}{\tau^2} = \frac{\mathrm{var}\, n}{\tau} \ , \tag{16}$$

so

$$\mathrm{var}\, n = \tau\, \overline{X_\tau^2} \ . \tag{17}$$

Now applying the Wiener-Khintchine theorem yields

$$S_n(0) = \lim_{\tau \to \infty} 2\tau\, \overline{X_\tau^2} = 2\, \mathrm{var}\, n \ . \tag{18}$$

To convert to current we prove Schottky's theorem. The spectral intensity of the fluctuating current $I(t)$ of average $\bar{I}$ is $S_I(0) = 2q\bar{I}$.

Proof: Poisson statistics apply, therefore $\mathrm{var}\, n = \bar{n}$. To get current from carrier flux, multiply by the carrier charge $q$. So

$$I(t) = qn(t) \qquad \text{Hence} \ \ \bar{I} = q\bar{n} \ . \tag{19}$$

Therefore

$$S_I(0) = q^2 S_n(0) = 2q^2 \mathrm{var}\, n = 2q^2\, \bar{n} = 2q\bar{I} \ , \tag{20}$$

where $S_I(0)$ is the spectral density of the current fluctuations.